2011

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State what it means for a real sequence to converge.

We say that the sequence $\langle x_n \rangle$ converges to l (and write $\lim x_n = l$) if, given $\epsilon > 0$ we can find a number N, such that, whenever $n > N$, we have $|x_n - l| < \epsilon$.

(or $\forall \epsilon > 0$, $\exists N$ such that $n > N \Longrightarrow |x_n - l| < \epsilon$.)

(b) Use the definition of convergence (not the combination theorem or any other theorems) to prove that

$$\lim_{n\to\infty} \frac{n^2+1}{2+3n^2} = \frac{1}{3}.$$

We have

$$\frac{n^2+1}{3n^2+2} - \frac{1}{3} = \frac{3n^2+3}{3(3n^2+2)} - \frac{3n^2+2}{3(3n^2+2)} = \frac{3n^2+3-3n^2-2}{3(3n^2+2)} = \frac{1}{3(3n^2+2)}.$$

To make $|x_n - 1/3| < \epsilon$, we need to have

$$\left|\frac{1}{3(3n^2+2)}\right| < \epsilon \Leftrightarrow 3n^2+2 > \frac{1}{3\epsilon}$$

$$\Leftrightarrow n^2 > \frac{1}{3}\left(\frac{1}{3\epsilon} - 2\right) \Leftrightarrow n > \sqrt{\frac{1}{9\epsilon} - \frac{2}{3}}.$$

So we take

$$N = \sqrt{\frac{1}{9\epsilon} - \frac{2}{3}},$$

provided that the expression under the square root is nonnegative. If it is negative the inequality $n^2 > \frac{1}{3}\left(\frac{1}{3\epsilon} - 2\right)$ is satisfied for all natural numbers, so we can take $N = 1$.

(c) State what it means for a real sequence to be a Cauchy sequence.

We say that the sequence $\langle x_n \rangle$ is Cauchy, if, given $\epsilon > 0$ we can find a number N, such that, whenever $n > N$, and $m > N$, we have $|x_n - x_m| < \epsilon$.

(or $\forall \epsilon > 0$, $\exists N$ such that $n > N$ and $m > N \Longrightarrow |x_n - x_m| < \epsilon$.)

(d) Use the definition (not a theorem) to show that the sequence $\langle a_n \rangle$ given by

$$a_n = \frac{1}{n}$$

is a Cauchy sequence.

Given $\epsilon > 0$ we must find N such that

$$n > N, \quad m > N \Longrightarrow |a_n - a_m| < \epsilon.$$

Since $a_n = 1/n$ and $a_m = 1/m$, we investigate

$$|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$$

by the triangle inequality. Since $n > N \Leftrightarrow 1/n < 1/N$ and $m > N \Leftrightarrow 1/m < 1/N$, we have

$$n > N, \quad m > N \Longrightarrow |a_n - a_m| \leq \frac{1}{n} + \frac{1}{m} < \frac{1}{N} + \frac{1}{N} = \frac{2}{N}.$$

To guarantee $|a_n - a_m| < \epsilon$, it suffices to have

$$\frac{2}{N} \leq \epsilon \Leftrightarrow N \geq \frac{2}{\epsilon}.$$

We take, therefore,

$$N = \frac{2}{\epsilon}.$$

(e) Prove that every convergent sequence of reals is a Cauchy sequence.

We are given that the sequence has a limit, say l, i.e. $\lim_{n\to\infty} x_n = l$. This means: Given $\epsilon > 0$ we can find N such that

$$n > N \Longrightarrow |x_n - l| < \frac{\epsilon}{2}.$$

We are asked to prove that it is Cauchy, i.e. $\forall\epsilon > 0$, $\exists N$ such that $n > N$ and $m > N \Longrightarrow |x_n - x_m| < \epsilon$.

So, given $\epsilon > 0$ we use the same N as in the equation above to deduce that, whenever $n > N$ and $m > N$ we have at the same time $|x_n - l| < \epsilon/2$ and $|x_m - l| < \epsilon/2$. Then

$$|x_n - x_m| = |x_n - l + l - x_m| \leq |x_n - l| + |l - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

2. (a) State the definition of $\lim_{x\to a^+} f(x) = l$.

We say that the limit of $f(x)$ as x tends to a from the right is l, if, given $\epsilon > 0$ we can find a $\delta > 0$, such that: whenever $a < x < a + \delta$, we have $|f(x) - l| < \epsilon$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 3, & (x < 2), \\ 6/x, & (x \geq 2). \end{cases}$$

Show carefully (using ϵ and δ) that

$$\lim_{x \to 2^-} f(x) = 1, \quad \lim_{x \to 2^+} f(x) = 3.$$

(i) For $\lim\limits_{x \to 2-} f(x) = 1$ we need to prove

$$\forall \epsilon > 0 \quad \exists \delta > 0 : 2 - \delta < x < 2 \Longrightarrow |f(x) - 1| < \epsilon.$$

We can restrict our attention to $x > 0$ (and < 2). We have $f(x) = x^2 - 3 < 4 - 3 = 1$. Therefore, $|f(x) - 1| = 1 - f(x) = 1 - (x^3 - 3) = 4 - x^2$. This gives

$$|f(x) - 1| < \epsilon \Leftrightarrow 4 - x^2 < \epsilon \Leftrightarrow 4 - \epsilon < x^2.$$

If $\epsilon \leq 4$ we get

$$|f(x) - 1| < \epsilon \Leftrightarrow \sqrt{4 - \epsilon} < |x|.$$

Then we get

$$|f(x) - 1| < \epsilon \Leftrightarrow \sqrt{4 - \epsilon} < x.$$

We match

$$2 - \delta = \sqrt{4 - \epsilon} \Leftrightarrow \delta = 2 - \sqrt{4 - \epsilon}.$$

We only need to prove that

$$\delta > 0 \Leftrightarrow 2 - \sqrt{4 - \epsilon} > 0 \Leftrightarrow 2 > \sqrt{4 - \epsilon} \Leftrightarrow 4 > 4 - \epsilon \Leftrightarrow 0 > -\epsilon,$$

which is true. The case $\epsilon > 4$ is even easier: In this case $4 - x^2 < \epsilon$ is always true $(x^2 \geq 0)$ as

$$4 - x^2 \leq 4 < \epsilon.$$

This means we can take any $\delta > 0$ with $\delta < 2$ in this case. The condition $\delta < 2$ guarantees we are still working with $x > 0$.

(ii) We need to show that given $\epsilon > 0$, we can find $\delta > 0$ such that

$$2 < x < 2 + \delta \Longrightarrow |f(x) - 3| < \epsilon.$$

We have for $x > 2$

$$|f(x)-3| = \left|\frac{6}{x} - 3\right| = \frac{|6-3x|}{|x|} = \frac{3x-6}{x} < \frac{3x-6}{2} < \epsilon \Leftrightarrow 3x-6 < 2\epsilon \Leftrightarrow x < 2+\frac{2}{3}\epsilon.$$

So we can take $\delta = 2\epsilon/3$.

(c) Let f be continuous on the compact interval $[a, b]$. Show that f is bounded on $[a, b]$.

We will prove that f is bounded above. If it is not, given n, we can find a number $x_n \in [a, b]$ with $f(x_n) > n$. These numbers form a sequence $\langle x_n \rangle$, $n = 1, 2, \ldots$. This sequence is bounded, as all the terms are in $[a, b]$. By the Bolzano-Weierstrass theorem, it has a convergent subsequence, call it x_{n_r}, $r = 1, 2, \ldots$. Call its limit ξ. As $a \leq x_{n_r} \leq b$, we also have $a \leq \xi = \lim x_{n_r} \leq b$. By the continuity of $f(x)$ we have

$$f(\xi) = \lim f(x_{n_r}).$$

On the other hand, $f(x_{n_r}) > n_r \geq r \to \infty$. But this means that the sequence $f(x_{n_r})$ is unbounded, while it converges, which implies that it is bounded. This is a contradiction. So $f(x)$ is bounded above. For the bound below, we can use $-f(x)$. It is continuous on $[a, b]$ so it is bounded above by M, say. Then

$$-f(x) \leq M \Longrightarrow f(x) \geq -M, \quad \forall x \in [a, b],$$

i.e. $f(x)$ is bounded below.

(d) Can you apply the theorem in (c) to the function f in (b) on the interval $[0, 5]$? Determine (with explanation) whether the function f is bounded on $[0, 5]$ or not.

The function f is not continuous at 2, since the limits from the left and from the right are different. Consequently, we cannot apply the theorem in (c) to this function. This does not mean that the function is unbounded on $[0, 5]$. In fact it is. On $[0, 2)$ we have $f(x) = x^2 - 3$. Since $x^2 - 3$ is increasing on this interval, we have $-3 \leq x^2 \leq 1$. On the interval $[2, 5]$ we have $f(x) = 6/x$. Clearly this is a decreasing function, so that $3 \geq 6/x \geq 6/5$. So on $[0, 5]$ we have $-3 \leq f(x) \leq 3$.

3. (a) State and prove the sandwich theorem for sequences.

Theorem: Suppose that $y_n \to l$ as $n \to \infty$ and $z_n \to l$ as $n \to \infty$. If $y_n \leq x_n \leq z_n$, $n = 1, 2, \ldots$ then $x_n \to l$ as $n \to \infty$.

Proof: Let $\epsilon > 0$ be given. Since $y_n \to l$, we can find N_1 such that

$$n > N_1 \Longrightarrow |y_n - l| < \epsilon,$$

and the last inequality with absolute values is equivalent to

$$l - \epsilon < y_n < l + \epsilon.$$

Since $z_n \to l$, we can find N_2 such that

$$n > N_2 \Longrightarrow |z_n - l| < \epsilon,$$

and the last inequality with absolute values is equivalent to

$$l - \epsilon < z_n < l + \epsilon.$$

Let $N = \max(N_1, N_2)$. Then whenever $n > N$ we have both $n > N_1$ and $n > N_2$, so we can apply the inequalities above to get

$$l - \epsilon < y_n < l + \epsilon, \quad \text{and} \quad l - \epsilon < z_n < l + \epsilon.$$

This gives

$$l - \epsilon < y_n \leq x_n \leq z_n < l + \epsilon \Longrightarrow l - \epsilon < x_n < l + \epsilon.$$

So we have found N such that

$$n > N \Longrightarrow |x_n - l| < \epsilon.$$

Hence, $x_n \to l$ as $n \to \infty$.

(b) Show that $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n} = 5$.

(*Hint*: You may assume that $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for $a > 0$.)

We have that $3^n + 5^n < 5^n + 5^n = 2 \cdot 5^n$ for $n \in \mathbb{N}$. Obviously $3^n + 5^n > 5^n$. These imply

$$5 = \sqrt[n]{5^n} < \sqrt[n]{3^n + 5^n} < \sqrt[n]{2 \cdot 5^n} = \sqrt[n]{2} \cdot 5.$$

Since $\sqrt[n]{a} \to 1$, for $a > 0$, we deduce that

$$\lim \sqrt[n]{2} \cdot 5 = 1 \cdot 5 = 5.$$

Now we use the sandwich theorem, with $y_n = 5$, $x_n = \sqrt[n]{3^n + 5^n}$, $z_n = \sqrt[n]{2} \cdot 5$, and $y_n < x_n < z_n$ and conclude that

$$\lim \sqrt[n]{3^n + 5^n} = 5.$$

(c) Define what it means for the series $\sum_{n=1}^{\infty} a_n$ to converge.

We define the partial sums of the series as follows:

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \ldots \quad s_N = a_1 + a_2 + \cdots + a_N.$$

If the sequence of partial sums $\langle s_N \rangle$ is convergent and

$$\lim_{N\to\infty} s_N = s,$$

then we say that the series $\sum_{n=1}^{\infty} a_n$ converges and its sum is s.

(d) Determine with explanations whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} n^3 \left(\frac{1}{2}\right)^n, \qquad \sum_{n=1}^{\infty} \sqrt[n]{3^n + 5^n}.$$

For the first series the easiest test to use is the ratio test. We have $a_n = n^3 \left(\frac{1}{2}\right)^n$, $a_{n+1} = (n+1)^3 \left(\frac{1}{2}\right)^{n+1}$. These give

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n} = \frac{(n+1)^3 \left(\frac{1}{2}\right)^{n+1}}{n^3 \left(\frac{1}{2}\right)^n} = \left(\frac{n+1}{n}\right)^3 \frac{1}{2} = \left(1 + \frac{1}{n}\right)^3 \frac{1}{2} \to (1+0)^3 \frac{1}{2} = \frac{1}{2} < 1.$$

Since the limit is < 1, the ratio test implies that the series converges.

We proved that $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n} = 5 \neq 0$. By the N-th term test for divergence, the series $\sum_{n=1}^{\infty} \sqrt[n]{3^n + 5^n}$ does not converge.

4. (a) (i) State the Cauchy–Schwarz inequality.

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Then

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{1/2} \left(b_1^2 + b_2^2 + \cdots + b_n^2\right)^{1/2},$$

or

$$\left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \cdot \left(b_1^2 + b_2^2 + \cdots + b_n^2\right).$$

(ii) Let $x_1, x_2, \ldots, x_n$ and $w_1, w_2, \ldots, w_n$ be positive numbers with $\sum_{j=1}^{n} w_j^2 = 1$. Use the Cauchy–Schwarz inequality to show that

$$\left(\sum_{j=1}^{n} x_j \cdot w_j^2\right)^2 \leq \sum_{j=1}^{n} (x_j^2 \cdot w_j^2).$$

We take

$$a_j = x_j w_j, \quad b_j = w_j, \quad j = 1, 2, \ldots, n$$

in the Cauchy–Schwarz inequality. We get

$$(x_1 w_1 \cdot w_1 + x_2 w_2 \cdot w_2 + \cdots + x_n w_n \cdot w_n)^2$$

$$\leq \left(x_1^2 w_1^2 + x_2^2 w_2^2 + \cdots + x_n^2 w_n^2\right) \cdot (w_1^2 + w_2^2 + \cdots + w_n^2) = \sum_{j=1}^{n} x_j^2 w_j^2,$$

using the given condition $w_1^2 + w_2^2 + \cdots + w_n^2 = 1$. The left-hand side is exactly the left-hand side of the required inequality:

$$\left(\sum_{j=1}^{n} x_j w_j^2\right)^2 = (x_1 w_1 \cdot w_1 + x_2 w_2 \cdot w_2 + \cdots + x_n w_n \cdot w_n)^2.$$

(iii) If the series $\sum_{n=1}^{\infty} a_n^2$ converges, show that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{3/2}}$$

converges absolutely.

We need to prove that $\sum_{n=1}^{\infty} \left|\frac{a_n}{n^{3/2}}\right|$ converges. We consider its partial sums

$$\sum_{n=1}^{N} \left|\frac{a_n}{n^{3/2}}\right|$$

and prove that they are bounded above. We apply the Cauchy-Schwarz inequality to get

$$\sum_{n=1}^{N}\left|\frac{a_n}{n^{3/2}}\right| \leq \left(\sum_{n=1}^{N} a_n^2 \cdot \sum_{n=1}^{N}\frac{1}{n^3}\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} a_n^2 \cdot \sum_{n=1}^{\infty}\frac{1}{n^3}\right)^{1/2}.$$

The first infinite series converges by assumption. For the second we notice that it is $\zeta(3)$, i.e. $s = 3 > 1$. As a result it converges. The right-hand side is a fixed number (not infinity) and the partial sums are bounded above. They form an increasing sequence bounded above, so the series $\sum_{n=1}^{\infty}\left|\frac{a_n}{n^{3/2}}\right|$ converges.

(b) State and prove the Bolzano–Weierstrass Theorem. You may assume that every sequence of reals has a monotone subsequence.

Theorem: (Bolzano–Weierstrass) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof: Let $\langle x_n\rangle$ be a bounded sequence. It has a monotone subsequence, say $\langle x_{n_r}\rangle$. Since the whole sequence $\langle x_n\rangle$ is bounded, the subsequence $\langle x_{n_r}\rangle$ is also bounded. So $\langle x_{n_r}\rangle$ is a monotone and bounded sequence. Such a sequence converges to its supremum (if it is increasing), or to its infimum (if it is decreasing).

5. (a) State the Intermediate Value Theorem.

Theorem. Let $f : [a, b] \to \mathbf{R}$ be continuous. If λ lies between $f(a)$ and $f(b)$, then we can find a ξ between a and b such that $\lambda = f(\xi)$.

(b) Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$.

Prove that for some $\xi \in [0,1]$ we have $f(\xi) = \xi$.

Define

$$g : [0,1] \to \mathbf{R}, \quad g(x) = f(x) - x.$$

Then g is a continuous function on $[0,1]$. Moreover,

$$g(0) = f(0) - 0 = f(0) \geq 0, \quad \text{as } f(0) \geq 0,$$

$$g(1) = f(1) - 1 = f(1) - 1 \leq 0, \qquad \text{as } f(1) \leq 1.$$

If $g(0) = 0$, then we take $\xi = 0$, as $f(0) = 0$. If $g(1) = 0$, then we take $\xi = 1$, as $f(1) = 1$.

If $g(0) \neq 0$ and $g(1) \neq 0$ we apply the intermediate value theorem to g on the interval $[0,1]$ with $\lambda = 0$, as

$$g(0) > 0 > g(1).$$

The result is that we can find a $\xi \in (0,1)$ with $g(\xi) = 0$, i.e. $f(\xi) - \xi = 0$, i.e. $f(\xi) = \xi$.

(c) Let y be positive. Using the function $f(x) = x^2$, show that y has a square root.

Case 1: $0 < y < 1$. Let $\lambda = y$ in the Intermediate Value Theorem. Take $a = 0$, $b = 1$. Since $f(0) = 0$ and $f(1) = 1$, we have $f(0) < \lambda < f(1)$. The intermediate value theorem provides a $\xi \in (0,1)$ with $f(\xi) = y \Leftrightarrow \xi^2 = y$, i.e. $\xi = \sqrt{y}$.

Case 2: $y = 1$. Obviously $\sqrt{1} = 1$.

Case 3: $y > 1$. Then $y^2 > y$. We apply the Intermediate Value theorem on the interval $[a,b] = [1,y]$ with $\lambda = y$. We see that $f(1) < \lambda < f(y) = y^2$. The Theorem provides a $\xi \in (1,y)$ with $f(\xi) = y \Leftrightarrow \xi^2 = y$, i.e. $\xi = \sqrt{y}$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which

$$(f(x))^2 - x^2 = 1, \quad \forall x \in \mathbb{R},$$

and $f(0) = -1$. Show that

$$f(x) = -\sqrt{1+x^2}, \quad \forall x \in \mathbb{R}.$$

We solve the equation $(f(x))^2 - x^2 = 1$ easily to get

$$f(x) = \pm\sqrt{1+x^2}.$$

We have to show that for all $x \in \mathbb{R}$ the correct sign to take is $-$. We first check that it is correct for $x = 0$. We are given $f(0) = -1$. Since $\sqrt{1+0^2} = 1$, the correct sign is $-$ for $x = 0$. Assume that for some x_0 we have

$$f(x_0) = \sqrt{1+x_0^2}.$$

Obviously $1 + x_0^2 > 1$, so that $\sqrt{1+x_0^2} \geq 1$ and $f(x_0) \geq 1$. By the intermediate value theorem applied to the interval between x_0 and 0, i.e. $[x_0, 0]$, if $x_0 < 0$ and $[0, x_0]$, if $x_0 > 0$, with $\lambda = 0$ we can find a ξ with

$$f(\xi) = 0.$$

This gives $f(\xi)^2 - \xi^2 = 0^2 - \xi^2$, while the given equation with $x = \xi$ gives $f(\xi)^2 - \xi^2 = 1$. Therefore, $-\xi^2 = 1 \Leftrightarrow \xi^2 = -1 < 0$, which is a contradiction, as squares are nonnegative. Therefore there is no x_0 with $f(x_0) = \sqrt{1+x_0^2}$.

6. (a) (i) Show that for all positive numbers y we have

$$\ln(y) \leq y - 1.$$

You may assume that $e^x \geq x + 1$ for all $x \in \mathbb{R}$.

We plug $x = \ln(y)$ into $e^x \geq x + 1$ to get

$$e^{\ln(y)} \geq \ln(y) + 1 \Leftrightarrow y \geq \ln(y) + 1 \Leftrightarrow y - 1 \geq \ln(y).$$

(ii) State and prove the Arithmetic Mean – Geometric Mean Inequality for n nonnegative numbers $a_1, a_2, \ldots, a_n$. You may use (i).

AM–GM:

$$\mathrm{GM} = \sqrt[n]{a_1 \cdot a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} = \mathrm{AM}.$$

Naturally the geometric mean can also be written as $\mathrm{GM} = (a_1 \cdot a_2 \cdots a_n)^{1/n}$.

Proof

Case 1: For all $i = 1, 2, \ldots, n$ we have $a_i = 0$. Then $\mathrm{AM} = 0 = \mathrm{GM}$ so the inequality is true.

Case 2: At least one of the nonnegative numbers is nonzero. This implies that the arithmetic mean, which we denote by A, is nonzero. We apply (i) successively to the numbers a_i/A, for $i = 1, 2, \ldots, n$. We get

$$\begin{aligned} \ln\left(\frac{a_1}{A}\right) &\leq \frac{a_1}{A} - 1, \\ \ln\left(\frac{a_2}{A}\right) &\leq \frac{a_2}{A} - 1, \\ &\vdots \\ \ln\left(\frac{a_n}{A}\right) &\leq \frac{a_n}{A} - 1. \end{aligned}$$

We sum up the inequalities above to get

$$\sum_{i=1}^{n} \ln\left(\frac{a_i}{A}\right) \leq \sum_{i=1}^{n} \frac{a_i}{A} - n = \frac{\sum_{i=1}^{n} a_i}{A} - n = \frac{nA}{A} - n = 0.$$

Using the property $\ln(ab) = \ln(a) + \ln(b)$ we get

$$\ln\left(\prod_{i=1}^{n} \frac{a_i}{A}\right) \leq 0 \Longrightarrow \prod_{i=1}^{n} \frac{a_i}{A} \leq 1,$$

as ln is an increasing function and $\ln 1 = 0$. We deduce

$$\frac{\prod_{i=1}^{n} a_i}{A^n} \leq 1 \Longrightarrow \prod_{i=1}^{n} a_i \leq A^n \Longrightarrow \left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq A \Leftrightarrow \mathrm{GM} \leq \mathrm{AM}.$$

(b) (i) For $0 < x < y$ prove the following inequalities

$$x < \frac{2}{\frac{1}{x} + \frac{1}{y}} < \sqrt{xy} < y.$$

Remark: The middle inequality says that the harmonic mean of x and y is less than their geometric mean. Both means are between the two numbers, this is the content of the other two inequalities. We prove them with strict inequalities for two distinct numbers x, y, $x < y$.

$$\sqrt{xy} < y \Leftrightarrow xy < y^2 \Leftrightarrow x < y$$

by squaring inequalities with positive terms, and canceling $y > 0$.

$$x < \frac{2}{1/x + 1/y} \Leftrightarrow x < \frac{2}{(y+x)/xy} \Leftrightarrow x < \frac{2xy}{x+y} \Leftrightarrow x(x+y) < 2xy \Leftrightarrow x+y < 2y \Leftrightarrow x < y,$$

where we have multiplied with the positive number $x + y$ and cancelled the positive number x.

$$\frac{2}{1/x + 1/y} < \sqrt{xy} \Leftrightarrow \frac{2xy}{x+y} < \sqrt{xy} \Leftrightarrow \frac{2xy}{\sqrt{xy}} < x + y$$

$\Leftrightarrow 2\sqrt{xy} < x + y \Leftrightarrow 4xy < (x+y)^2 = x^2 + y^2 + 2xy \Leftrightarrow 0 < x^2 + y^2 - 2xy = (x-y)^2.$

The last inequality is clear, since $x \neq y$, so the square of $x - y$ is positive.

(ii) Define the sequence $\langle x_n \rangle$ by

$$x_1 = 1/2, \quad x_2 = 1, \quad x_{2n+1} = \sqrt{x_{2n}x_{2n-1}}, \quad x_{2n+2} = \frac{2}{\frac{1}{x_{2n}} + \frac{1}{x_{2n+1}}}, \quad n \geq 1.$$

Use induction and (i) to prove that

$$x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}, \quad n \in \mathbb{N}.$$

Deduce that the subsequences given by $\langle x_{2n} \rangle$ and $\langle x_{2n-1} \rangle$ converge to the same limit. What does this imply for the sequence $\langle x_n \rangle$?

We compute $x_3 = \sqrt{x_2 x_1} = \sqrt{1/2}$ and

$$x_4 = \frac{2}{1/x_2 + 1/x_3} = \frac{2}{1 + \sqrt{2}}.$$

We show

$$P(n): \quad x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}$$

by induction.

$$P(1): \quad x_1 < x_3 < x_4 < x_2 \Leftrightarrow 1/2 < \sqrt{1/2} < \frac{2}{1+\sqrt{2}} < 1.$$

We check:

$$\frac{1}{2} < \sqrt{1/2} \Leftrightarrow 2 > \sqrt{2} \Leftrightarrow 4 > 2$$

which is true.

$$\sqrt{1/2} < \frac{2}{1+\sqrt{2}} \Leftrightarrow \sqrt{2} > \frac{1+\sqrt{2}}{2} \Leftrightarrow 2\sqrt{2} > 1+\sqrt{2} \Leftrightarrow \sqrt{2} > 1,$$

which is true.

$$\frac{2}{1+\sqrt{2}} < 1 \Leftrightarrow 1+\sqrt{2} > 2 \Leftrightarrow \sqrt{2} > 1,$$

which is true.

Assume $P(n)$ is true. We need to show that $P(n+1)$ is true. We have

$$P(n+1): \quad x_{2n+1} < x_{2n+3} < x_{2n+4} < x_{2n+2}$$

We use $x = x_{2n+1}$ and $y = x_{2n+2}$ in (i) together with $x_{2n+1} < x_{2n+2}$ from $P(n)$ to get

$$x_{2n+1} < \sqrt{x_{2n+1}x_{2n+2}} = x_{2n+3} < x_{2n+2}.$$

Since

$$x_{2n+4} = \frac{2}{1/x_{2n+2} + 1/x_{2n+3}},$$

we use (i) with $x = x_{2n+3}$ and $y = x_{2n+2}$ and the fact just proved that $x_{2n+3} < x_{2n+2}$ to deduce:

$$x_{2n+3} < \frac{2}{1/x_{2n+2} + 1/x_{2n+3}} = x_{2n+4} < x_{2n+2}.$$

This shows that $P(n+1)$ is true and completes the induction.

The subsequence with odd subscripts is strictly increasing and bounded above by x_2, so it converges to its least upper bound. The subsequence with even subscripts is strictly decreasing and bounded below by x_1, so it converges to its infimum. Say

$$\lim_n x_{2n} = l, \quad \lim_n x_{2n-1} = m.$$

We need to prove that $l = m$. We look at $x_{2n+2} = \sqrt{x_{2n}x_{2n-1}}$ and take limits of both sides. Since $x_{2n} \to l$, we also have $x_{2n+2} \to l$, as a subsequence. We get

$$l = \lim x_{2n+2} = \sqrt{\lim x_{2n} \lim x_{2n-1}} = \sqrt{lm} \Longrightarrow l = \sqrt{lm} \Longrightarrow l^2 = lm \Longrightarrow l = m$$

by canceling l. This is justified as $l \geq x_1 = 1/2 > 0$. As both subsequences with even and odd subscripts converge to the same limit, the whole sequence converges to this limit.